# MA114 Summer 2018
## Worksheet 25 – Calculus with Polar Curves – 7/26/18

$$\frac{dy}{dx} = \frac{-2\sin^2\theta + 2\cos^2\theta + \cos\theta}{-2\sin 2\theta - \sin\theta}$$

1. Find the slope $dy/dx$ for the following polar curves:

    a) $r = 2\cos(\theta) + 1$

    b) $r = \dfrac{1}{\theta}$

a) $x = (2\cos\theta + 1)\cos\theta$

$y = (2\cos\theta + 1)\sin\theta$

$x' = -2\sin\theta\cos\theta - (2\cos\theta + 1)\sin\theta$

$y' = -2\sin\theta\sin\theta + (2\cos\theta + 1)\cos\theta$

2. Compute the slope of the tangent line to the graph of $r = \sin\theta$ at $\theta = \pi/3$, and sketch the curve and the tangent line.

3. Find the area enclosed by one leaf of the curve $r = \sin 2\theta$.

4. Find the arc length of one leaf of the curve $r = \sin 2\theta$.

5. Find the area between the inner and outer loop of the limaçon $r = 2\cos\theta - 1$.

6. Find the tangent line to the polar curve $r = \theta^2$ at $\theta = \pi$.

7. Find the length of the curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

8. Find the area of the region that lies inside both the curves $r = \sqrt{3}\sin\theta$ and $r = \cos\theta$.

9. Find the point(s) where the tangent line to the curve $r = 2 + \sin\theta$ is horizontal.

b) $x = \frac{1}{\theta}\cos\theta$

$y = \frac{1}{\theta}\sin\theta$

$x' = -\dfrac{\cos\theta}{\theta^2} - \dfrac{\sin\theta}{\theta}$

$y' = -\dfrac{\sin\theta}{\theta^2} + \dfrac{\cos\theta}{\theta}$

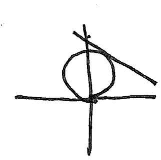
$$\frac{dy}{dx} = \frac{\dfrac{-\sin\theta + \theta\cos\theta}{\theta^2}}{\dfrac{-\cos\theta - \theta\sin\theta}{\theta^2}} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

2) $x = \sin\theta\cos\theta$ $\quad x' = \cos^2\theta - \sin^2\theta = \cos 2\theta$

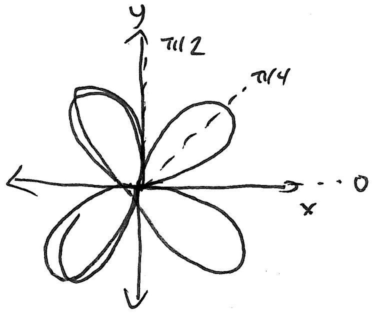
$y = \sin^2\theta$ $\quad y' = 2\sin\theta\cos\theta = \sin 2\theta$

$$\frac{dy}{dx} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

At $\theta = \frac{\pi}{3}$: $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$



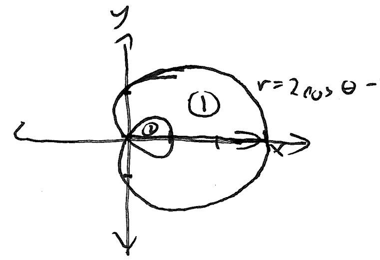
3)




$$A = \int_0^{\pi/2} \frac{1}{2}\left(\sin(2\theta)\right)^2 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$
$$= \frac{1}{4}\int_0^{\pi/2} 1 - \cos(4\theta)\, d\theta$$
$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi/2}$$
$$= \frac{\pi}{8}$$

4)
$$s = \int_0^{\pi/2} \sqrt{r^2 + (r')^2}\, d\theta$$
$$= \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}\, d\theta$$
$$= \int_0^{\pi/2} \sqrt{1 + 3\cos^2(2\theta)}\, d\theta$$
$\approx 9.68845$ (by computer)

5)




$$\text{Area} = 2\,(\text{Area }1 - \text{Area }2)$$
$$= 2\left(-\int_0^{\pi/3} \frac{1}{2}(2\cos\theta - 1)^2 d\theta + \int_\pi^{5\pi/3} \frac{1}{2}(2\cos\theta - 1)^2 d\theta\right)$$
$$= \cancel{2}\left(\int_\pi^{5\pi/3} 4\cos^2\theta - 4\cos\theta + 1\, d\theta - \int_0^{\pi/3} 4\cos^2\theta + 4\cos\theta + 1\, d\theta\right)$$
$$= \int_\pi^{5\pi/3} (2 + 2\cos 2\theta - 4\cos\theta + 1)\, d\theta - \int_0^{\pi/3} (2 + 2\cos 2\theta - 4\cos\theta + 1)\, d\theta$$
$$= 3\theta + \sin 2\theta - 4\sin\theta\Big|_\pi^{5\pi/3} - \left(3\theta + \sin 2\theta - 4\sin\theta\Big|_0^{\pi/3}\right)$$
$$= 5\pi + \sin\left(\tfrac{10\pi}{3}\right) - 4\sin(5\pi/3) - 3\pi - \left(\pi + \sin\left(\tfrac{2\pi}{3}\right) - 4\sin\left(\tfrac{\pi}{3}\right)\right)$$
$$= \pi + \left(-\frac{\sqrt{3}}{2}\right) - 4\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} + 4\left(\frac{\sqrt{3}}{2}\right)$$
$$= \boxed{\pi + 3\sqrt{3}}$$

6) $r = \theta^2$

$x = \theta^2\cos\theta \qquad dx/d\theta = 2\theta\cos\theta - \theta^2\sin\theta$

$y = \theta^2\sin\theta \qquad dy/d\theta = 2\theta\sin\theta + \theta^2\cos\theta$

At $\theta = \pi$: $dx/d\theta = -2\pi$, $dy/d\theta = -\pi^2$

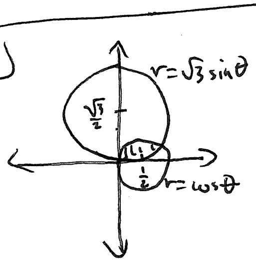
So $dy/dx = \frac{-\pi^2}{-2\pi} = \frac{\pi}{2}$ at $\theta = \pi$.

Then $x(\pi) = -\pi^2$, $y(\pi) = 0$, so the tangent line is

$$\boxed{y - 0 = \frac{\pi}{2}(x + \pi^2)}$$

7) $r = \theta^2$.
$$s = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta$$
$$= \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta \qquad u = \theta^2 + 4$$
$$= \frac{1}{2}\left(\sqrt{u}\right)^3\Big|_4^{4\pi^2+4}$$
$$= \boxed{\frac{1}{2}\left((4\pi^2+4)^{3/2} - 8\right)}$$

8)




$\sqrt{3}\sin\theta = \cos\theta \iff \tan\theta = \frac{1}{\sqrt{3}}$

$\iff \theta = \frac{\pi}{6}$.

$$\text{So } A = \int_0^{\pi/6} \frac{1}{2}(\sqrt{3}\sin\theta)^2 d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(\cos\theta)^2 d\theta$$
$$= \frac{3}{2}\int_0^{\pi/6} \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$$
$$= \frac{3}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} + \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/6}^{\pi/2}$$
$$= \frac{3}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{1}{4}\left(\frac{\pi}{2} + 0 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \boxed{\frac{5\pi}{12} - \frac{\sqrt{3}}{4}}$$

8) $r = 2 + \sin\theta$

$x = 2\cos\theta + \sin\theta\cos\theta$

$y = 2\sin\theta + \sin^2\theta$

$dx/d\theta = -2\sin\theta + \cos^2\theta - \sin^2\theta$

$dy/d\theta = 2\cos\theta + 2\sin\theta\cos\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta + \sin(2\theta)}{-2\sin\theta + \cos(2\theta)}$$

if $\frac{dy}{dx} = 0$:

$2\cos\theta + \sin(2\theta) = 0$

$\theta = \pm\frac{\pi}{2}$

$2\cos\theta(1 + \sin\theta) = 0$

so $\cos\theta = 0$: $\theta = \pm\frac{\pi}{2}$

or $1 + \sin\theta = 0$: $\theta = -\frac{\pi}{2}$

Check: $\frac{dx}{d\theta}\left(\frac{\pi}{2}\right) = -3 \neq 0$

$\frac{dx}{d\theta}\left(-\frac{\pi}{2}\right) = 1 \neq 0$

So $(r, \theta)$: $\boxed{(3, \tfrac{\pi}{2}) \text{ and } (1, -\tfrac{\pi}{2})}$